

Transport in the XX chain at zero temperature: Emergence of flat magnetization profiles

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We study the connection between magnetization transport and magnetization profiles in zero-temperature XX chains. The time evolution of the transverse magnetization $m(x,t)$ is calculated using an inhomogeneous initial state that is the ground state at fixed magnetization but with m reversed from $-m_0$ for $x < 0$ to m_0 for $x > 0$. In the long-time limit, the magnetization evolves into a scaling form $m(x,t) = \Phi(x/t)$ and the profile develops a flat part ($m = \Phi = 0$) in the $|x/t| \leq c(m_0)$ region. The flat region shrinks to zero if $m_0 \rightarrow 1/2$ while it expands with the maximum velocity $c_0 = 1$ for $m_0 \rightarrow 0$. The states emerging in the scaling limit are compared to those of a homogeneous system where the same magnetization current is driven by a bulk field, and we find that the expectation values of various quantities (energy, occupation number in the fermionic representation) agree in the two systems. [S1063-651X(99)03605-3]

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I. INTRODUCTION

Transport in integrable systems is anomalous. This means that a harmonic lattice (or a transverse Ising chain) cannot support an internal thermal gradient when the two ends of the system are kept at different temperatures [1,2]. The temperature profile that forms is flat everywhere apart from the neighborhood of the boundaries, and the energy flux is clearly not proportional to the temperature gradient inside the sample. The origin of this anomaly is the fact that currents in integrable systems (energy currents in the above examples) often emerge as integrals of motion thus causing the transport coefficients to be divergent or ill defined [3].

The existence of flat density (temperature) profiles in the presence of (energy) currents points to the important role played by the homogeneous, current-carrying states in integrable systems. In previous papers [4,5], we suggested that a method of constructing such states at zero temperature is to add to the Hamiltonian an appropriate current with a Lagrange multiplier and then find the ground state of the system. This method can be applied with relative ease to models such as the transverse Ising and isotropic XY (so called XX) models, and one can calculate the effect of energy current on experimentally measurable correlations.

Since there are physical systems that are rather well approximated as XX models [6], and since the effect of currents should be measurable in inelastic neutron scattering experiments [7], one should carefully examine the validity of the assumptions underlying the calculations of the correlations. The Lagrange multiplier method is based on the assumption that the homogeneous, current-carrying states are the same whether they were induced by boundary conditions or by bulk driving fields. This is not an obvious assumption and its validity should be checked by first solving the boundary condition problem and then comparing the results with those obtained from the Lagrange-multiplier method. Unfortunately, the execution of this program runs into difficulties since the boundary condition problem is nontrivial and we

cannot provide an exact solution for any of the interesting cases.

One can make some progress, however, in a closely related problem. Namely, current-carrying states generated by inhomogeneous initial conditions can be studied and one can ask whether those states were describable by the Lagrange multiplier method. In order to illuminate the new problem, let us imagine the original problem as finding the steady state of a brick with two of its opposite sides kept at different temperatures. In the new problem, we have two semi-infinite bricks at different temperatures and, at time zero, the bricks are joined at the ends. If the bricks were integrable systems then one would expect that the time-evolution would produce an interesting temperature profile with a flat part around the joining point and it would be meaningful to compare the profile as well as various local expectation values to those obtained from the Lagrange-multiplier method. This is what we shall do in this paper for the XX chain after making the following simplifications. First, we note that the transverse magnetization m is a conserved quantity in the XX model. Since this is a much simpler quantity to treat, we shall study the transport of m instead of the transport of the energy. Second, we restrict our study to the zero-temperature properties of the system. Third, we shall use a simple inhomogeneous initial state pasted together from two zero-temperature homogeneous pieces with $m(x < 0, 0) = m_0$ and $m(x > 0, 0) = -m_0$.

As a result of the above simplifications, the problem of time evolution becomes solvable and the formation of a flat magnetization profile can be observed. We find that both the magnetization profile and the fermionic two-point correlations agree with the results obtained from the Lagrange multiplier method, thus indicating that the method does work for obtaining the homogeneous current carrying states.

We start (Sec. II) by defining the system and constructing the initial state in terms of fermionic operators. Next, the time evolution of the magnetization is calculated and the scaling form $m(x,t) = \Phi(x/t)$ of the long-time limit is discussed (Sec. III). Finally, we show that the local properties

of the scaling limit can be described using the homogeneous, current carrying states obtained from the Lagrange multiplier method (Sec. IV).

II. THE MODEL AND THE CONSTRUCTION OF THE INITIAL STATE

The system we investigate is the XX model defined by the Hamiltonian

$$H^{xx} = \sum_n h_n^{xx} = -J \sum_n (s_n^x s_{n+1}^x + s_n^y s_{n+1}^y), \quad (1)$$

where the spins s_n^α ($\alpha = x, y, z$) are $1/2$ times the Pauli matrices situated at the sites of a one-dimensional periodic lattice [$n = 0, \pm 1, \dots, \pm(N-1), N; s_{N+1}^\alpha = s_{-N+1}^\alpha$]. The coupling J is set to $J = 1$ throughout this paper and $N \rightarrow \infty$ is assumed in most of our calculations.

The z component of the total magnetization $M^z = \sum_n s_n^z$ is conserved in this system and, using the continuity equation for M^z , a local magnetization current can be defined [5]:

$$j_n^M = s_n^y s_{n+1}^x - s_n^x s_{n+1}^y. \quad (2)$$

It can be easily shown that the global magnetization current $J^M = \sum_n j_n^M$ commutes with the Hamiltonian (1) and thus one can classify the eigenstates of H^{xx} by the global current flowing through the system. Below we shall study how the initial conditions can generate states that are locally homogeneous and carry a finite magnetization current.

The time evolution that follows from a given initial condition can be calculated for the XX model since, using a Jordan-Wigner transformation, this model can be transformed into a set of free fermions [8]

$$H^{xx} = - \sum_k \cos k b_k^\dagger b_k, \quad (3)$$

where b_k is the annihilation operator of a fermion of wave number k . It follows that the time evolution of the b_k operators is known $b_k(t) = b_k(0) \exp(it \cos k)$ and a Fourier transform provides us with the time dependence of a local Fermi operator c_n as a sum of Bessel functions of the first kind [9]

$$c_n(t) = \sum_j c_j(0) i^{j-n} J_{j-n}(t). \quad (4)$$

The above expression is the starting point for our calculations since we wish study the local transverse magnetization s_n^z that can be expressed through the local Fermi operators as

$$s_n^z = c_n^\dagger c_n - \frac{1}{2}. \quad (5)$$

Our aim is to find the time evolution of the expectation value of s_n^z

$$m(n, t) = \langle \varphi | s_n^z(t) | \varphi \rangle \quad (6)$$

using an initial state $|\varphi\rangle$ that is expected to produce a magnetization current. The simplest such state has a symmetric step in the initial magnetization

$$\langle \varphi | s_n^z(0) | \varphi \rangle = \begin{cases} m_0 & \text{for } -N < n \leq 0, \\ -m_0 & \text{for } 1 \leq n \leq N. \end{cases} \quad (7)$$

The above condition, however, does not specify the initial state uniquely. In order to define $|\varphi\rangle$ precisely, we shall construct the state of the $2N$ spins as a direct product of states of two systems of N spins. The left system contains the $n \leq 0$ spins while the right one is built from the rest ($n \geq 1$). In order to get close to a zero-temperature, current-carrying state we choose the lowest energy state under the above conditions (7). Namely, both half chains are chosen to be in their ground states with the magnetizations being m_0 and $-m_0$ on the left and right, respectively. Another reason for choosing such a state comes from trying to avoid possible energy currents in the system. In the $N \rightarrow \infty$ limit, the above state is a ground state of the model at fixed $m = \pm m_0$ and the two homogeneous sides have the same energy. Thus one expects that only magnetization current will flow in the course of the time evolution of the system.

A formal definition of $|\varphi\rangle$ with the above properties can be given as follows. Let us define annihilation operators on the left and right by the Fourier transforms of the fermionic operators c_n on the appropriate sides of the system:

$$L_k = \frac{1}{\sqrt{N}} \sum_{j=-L+1}^0 e^{ikj} c_j, \quad R_k = \frac{1}{\sqrt{N}} \sum_{j=1}^L e^{ikj} c_j. \quad (8)$$

As can be shown, these operators are also fermionic operators with the anticommutation relations $\{L_k, L_q^\dagger\} = \{R_k, R_q^\dagger\} = \delta_{kq}$, and the initial state $|\varphi\rangle$ can be written as

$$|\varphi\rangle = \prod_{k=-k_-}^{k_-} R_k^\dagger \prod_{k'= -k_+}^{k_+} L_{k'}^\dagger |\downarrow\rangle, \quad (9)$$

where $|\downarrow\rangle$ is the state with all spins down and $k_\pm = \pi(\frac{1}{2} \pm m_0)$. In the following $m_0 > 0$ will be assumed without restricting the generality of the arguments.

III. NUMERICAL AND ANALYTICAL EVALUATION OF THE TRANSVERSE MAGNETIZATION

Once the initial state $|\varphi\rangle$ is given, the time evolution of s_n^z is expressed through initial correlations

$$\langle \varphi | s_n^z(t) | \varphi \rangle = \sum_{j,l} i^{j-l} J_{j-n}(t) J_{l-n}(t) \langle \varphi | c_l^\dagger(0) c_j(0) | \varphi \rangle - \frac{1}{2}. \quad (10)$$

It follows now from the special construction of $|\varphi\rangle$ that the terms with j and l on different sides of the origin vanish. Furthermore, the terms where $l-j$ is odd cancel since the expectation value is real. The calculation of $\langle \varphi | c_l^\dagger(0) c_j(0) | \varphi \rangle$ is carried out by using the explicit form [Eqs. (9)] for $|\varphi\rangle$ and the result is as follows:

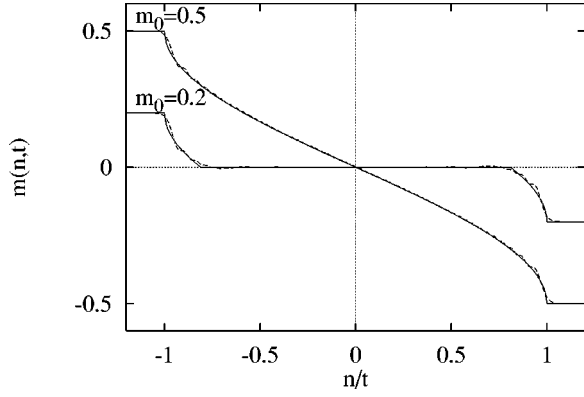


FIG. 1. Magnetization profile for initial conditions $m_0=0.5$ and $m_0=0.2$. The large-time limits approach the scaling curves (solid lines) given by Eq. (14).

$$i^{j-s} \langle \varphi | c_s^\dagger c_j | \varphi \rangle = \begin{cases} \pm \frac{\sin[\pi m_0(s-j)]}{\pi(s-j)} & \text{for } s-j \text{ even } \neq 0, \\ \frac{1}{2} \pm m_0 & \text{for } s-j=0. \end{cases} \quad (11)$$

The upper (lower) sign is valid for both s and j being in the left (right) part of the chain. Since the above expression depends only on the even integer $s-j$, the notation simplifies considerably by introducing $l=(s-j)/2$. Treating the $l=0$ term separately and using an identity [9] for the Bessel functions, $\sum_n J_n^2=1$, Eqs. (10) and (11) yield the following magnetization profile for $n \geq 1$:

$$m(n,t) = -m_0 \sum_{j=1-n}^{n-1} J_j^2(t) - 2 \sum_{l=1}^{\infty} \frac{\sin 2\pi m_0 l}{2\pi l} \sum_{j=1-n}^{n-1} J_j(t) J_{j+2l}(t). \quad (12)$$

The $n \leq 0$ values can also be calculated and they show the symmetry of the initial state, $m(n,t) = -m(-n+1,t)$. The above expression for $m(n,t)$ is the basis for both numerical and analytical results presented below.

A. Numerical results

The infinite sum (12) defining $m(n,t)$ converges well since the main contribution comes from terms with small l . The numerical evaluation of $m(n,t)$ poses a problem only for $m_0 \ll 1/2$ when the scaling limit $n \rightarrow \infty$, $t \rightarrow \infty$, with $n/t \sim 1$, cannot be readily reached due to the increasing number of terms that must be taken into account when evaluating the sums in Eq. (12).

The results from the numerical analysis (see Fig. 1) show that the magnetization profile evolves into a scaling form in the large-time limit

$$m(n,t) \approx \Phi(n/t) \quad (13)$$

with the following scaling function (analytically obtained in Sec. III B) giving an excellent fit to the data:

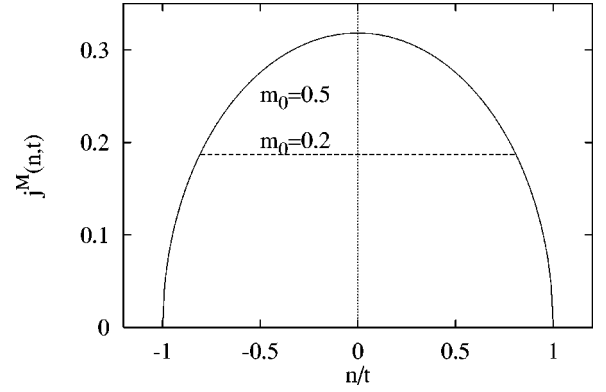


FIG. 2. Scaling functions for the magnetization-current for initial conditions $m_0=0.5$ and $m_0=0.2$ as given by Eq. (15).

$$\Phi(v) = -\Phi(-v) = \begin{cases} 0 & \text{for } 0 < v < \cos(\pi m_0) \\ -m_0 + \frac{1}{\pi} \arccos(v) & \text{for } \cos(\pi m_0) < v < 1 \\ -\frac{1}{2} & \text{for } 1 < v. \end{cases} \quad (14)$$

As one can see, the magnetization profile does develop a flat part that expands with a finite velocity $c = \cos \pi m_0$. The velocity $c(m_0)$ decreases with increasing m_0 and it diminishes for $m_0 \rightarrow 1/2$.

The conservation of magnetization implies that the magnetization current has a scaling form $j^M(n,t) \approx \Psi(n/t)$ as well, and the scaling function can be obtained from Eq. (14) using the continuity equation together with the boundary condition $j^M(n \rightarrow \pm \infty, t) = 0$. We find the scaling function

$$\Psi(v) = \Psi(-v) = \begin{cases} \frac{1}{\pi} \sin \pi m_0 & \text{for } 0 < v < \cos(\pi m_0) \\ \frac{1}{\pi} \sqrt{1-v^2} & \text{for } \cos(\pi m_0) < v < 1 \\ 0 & \text{for } 1 < v, \end{cases} \quad (15)$$

shown in Fig. 2.

B. Analytical results

Let us begin by deriving the analytical guess obtained from numerics (14) by treating the case $m_0=1/2$. This limiting case is especially simple since $\sin(2\pi m_0 l) = 0$ and hence one is left with the following expression:

$$m(n,t) = -\frac{1}{2} \sum_{j=1-n}^{n-1} J_j^2(t) = -\frac{1}{2} + \sum_{j \geq n} J_j^2(t). \quad (16)$$

We shall evaluate the scaling limit of the above sum by calculating the integral of the large n limit of the discrete derivative of $m(n,t)$ defined as [11]

$$\Phi'_n(v) = t[m(n+1, t) - m(n, t)]_{n/t=v} = -\frac{n}{v} J_n^2\left(\frac{n}{v}\right). \quad (17)$$

The cases $v > 1$ and $v < 1$ must be treated separately since the asymptotic properties of the Bessel functions change at $v = 1$ [10]. Namely, one finds

$$\Phi'_n(v) \sim \frac{1}{2\pi\sqrt{v^2-1}} \exp\left[-\frac{2n}{v}(v \cosh^{-1} v - \sqrt{v^2-1})\right], \quad v > 1 \quad (18)$$

while

$$\Phi'_n(v) \sim \frac{2}{\pi\sqrt{1-v^2}} \cos^2\left[\frac{n}{v}(\sqrt{1-v^2} - v \arccos v) - \frac{\pi}{4}\right], \quad v < 1. \quad (19)$$

As one can see, the derivative approaches zero exponentially for $v > 1$ (outside of the ‘‘light cone’’) and thus $\Phi(v)$ is constant in this region. The constant is determined by the boundary condition $\Phi(\infty) = -1/2$ and thus we have $\Phi(v) = -1/2$ for $v > 1$.

The case $v < 1$ is complicated by the fact that the limit $n \rightarrow \infty$ does not exist for $\Phi'_n(v)$ due to the increasing frequency of oscillations in Eq. (19). It should be noted, however, that we are interested in the integral of $\Phi'(v)$

$$\Phi(v) = \lim_{n \rightarrow \infty} \Phi_n(v) = \lim_{n \rightarrow \infty} \int_0^v \Phi'_n(y) dy, \quad (20)$$

and the $n \rightarrow \infty$ limit of the above integral does exist. The value of the integral can be obtained by replacing the fast oscillating \cos^2 term [see Eq. (19)] by its average value $1/2$. One then obtains

$$\Phi(v) = -\frac{1}{\pi} \arcsin v, \quad (21)$$

in agreement with Eq. (14) for $m_0 = 1/2$.

The evaluation of $\Phi(v)$ for $m_0 \neq 1/2$ is more involved but, essentially, it follows the above steps and yields the general expression Eq. (14). An outline of the calculation and some intermediate results are given in Appendix A.

IV. COMPARISON WITH THE LAGRANGE MULTIPLIER METHOD

A consequence of the emergence of scaling $m(n, t) \approx \Phi(n/t)$ is that the magnetization becomes locally flat, $m(n+1, t) - m(n, t) \sim t^{-1} \Phi' \rightarrow 0$, in the scaling limit. Since these locally homogeneous segments of the system become arbitrarily large for $t \rightarrow \infty$, one would like to ask if they could be described as a homogeneous, current-carrying state of an infinite chain as obtained from the Lagrange multiplier method [5].

In order to formulate the question more precisely, let us note that, for a given $n/t = v$ and in the neighborhood of n , the scaling state of the system may be characterized by the

local values of the conserved quantities and their fluxes, where the expectation value of a local operator $a_n(t)$ is defined in the scaling limit as

$$\langle a \rangle_v = \lim_{n, t \rightarrow \infty} \langle \varphi | a_n(t) | \varphi \rangle_{n/t=v}. \quad (22)$$

The simplest conserved quantities are the magnetization, $m = \langle s^z \rangle_v = \Phi(v)$ and the energy density $\varepsilon = \langle h^{xx} \rangle_v$ with the corresponding fluxes being the magnetization current $j^M = \langle j^M \rangle_v$ and the internal energy flux $j^I = \langle j^I \rangle_v$, where

$$J^I = \sum_n j_n^I = \sum_n s_n^z (s_{n-1}^y s_{n+1}^x - s_{n-1}^x s_{n+1}^y). \quad (23)$$

The energy density is homogeneous in the initial state (apart from the local perturbation of the domain wall at the origin) and it can be shown that this homogeneity remains intact in the long-time limit. Thus ε is determined by the initial condition and, in the scaling limit, can be expressed through m and j^M

$$\begin{aligned} \varepsilon &= \varepsilon(m, j^M) = -\frac{1}{\pi} \cos(\pi m_0) \\ &= -\frac{1}{\pi} \cos[\pi|m| + \arcsin(\pi j^M)]. \end{aligned} \quad (24)$$

Note that the homogeneity of the energy density implies a vanishing energy flux, $j^I = 0$, for all values of v and so Eq. (24) can be viewed as an equation of state connecting the relevant densities and fluxes.

It is clear that $(m, j^M, \varepsilon, j^I)$ gives only a partial characterization of the state since there are infinitely many conservation laws in the XX model. Nevertheless, one may assume that $(m, j^M, \varepsilon, j^I)$ were sufficient to describe *the states emerging in the scaling limit*. Then one can ask if the properties of these states were the same as those of the homogeneous, current-carrying state of an infinite system where the values of m, j^M, j^I were fixed by introducing conjugate fields (Lagrange multipliers) and by finding the ground state $|\psi\rangle$ of the following Hamiltonian:

$$\mathcal{H} = H^{xx} - hM^z - \lambda_M J^M - \lambda_I J^I. \quad (25)$$

The reason for this identification is that the scaling emerges as an evolution from an initial state of minimal energy at a given $m = m_0$. We assume here that local states of the scaling limit remain zero temperature states in the sense that they have minimal energy density at the given values of m, j^M , and j^I .

Since m, j^M, j^I are set by adjusting the Lagrange multipliers, a necessary condition for $|\psi\rangle$ having the local properties of the scaling states is the equality of its energy density as a function of m, j^M, j^I to the energy density ε given by Eq. (24). Verification of this equality can be viewed as checking whether the equation of states of the two systems were identical.

In order to obtain $\langle \psi | h_n^{xx} | \psi \rangle$, we use the results from our previous work [5] where the problem of energy current in the

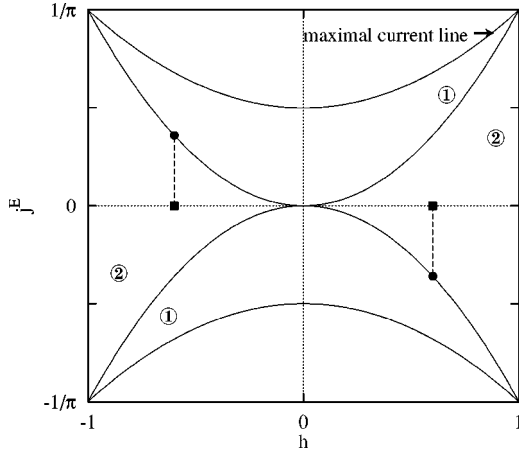


FIG. 3. Magnetic-field (h)-energy-flux ($j^E = j^I - h j^M$) phase diagram of an XX model where the energy flux is driven homogeneously in the bulk [5]. The dashed lines in region ② correspond to states of constant energy with the location of the dashed lines determined by the initial conditions. The two filled squares represent the $m = \pm m_0$ regions while the filled circles describe identical states corresponding to the plateau ($m=0$) in Fig. 1. Moving along the dashed lines provides the local states ($m, j^M, j^I=0$) in the transient region.

transverse XX model was investigated. In that work, we studied the ground-state properties of the following Hamiltonian:

$$\mathcal{H} = H^{xx} - h M^z - \lambda_E J^E \quad (26)$$

with the total energy flux J^E being a sum of the fluxes of internal and of magnetic energy

$$J^E = J^I - h J^M. \quad (27)$$

We found that the j^E, h phase diagram of this model has a region (called region ② in Fig.1 of [5], see Fig. 3 of the present paper) where the ground-state expectation value of the flux of internal energy is zero ($\langle \psi | J^I | \psi \rangle = 0$) and so the $j^I=0$ condition is automatically satisfied. The magnetization flux in region ② has been calculated in [5]. For the case of $h\lambda_E < 0$ that is needed in the setup we use ($m_0 > 0$), it has the form

$$j^M = -\frac{j^E}{h} = \frac{1}{\pi} \sqrt{1 - \lambda_E^{-2}} \quad (28)$$

and the calculation of the densities of magnetization and of energy is also a simple exercise with the following results:

$$m = \frac{1}{\pi} (-\arcsin \lambda_E^{-1} + \arcsin h) - \frac{1}{2} \operatorname{sgn}(h), \quad (29)$$

$$\langle \psi | h_n^{xx} | \psi \rangle = -\frac{1}{\pi} \sqrt{1 - h^2}. \quad (30)$$

Eliminating h and λ_E from Eqs. (28), (29), and (30) produces then Eq. (24), i.e., $\langle \psi | h_n^{xx} | \psi \rangle = \varepsilon(m, j^M)$. This result indicates that the local states derived as a scaling limit can indeed be obtained with the help of the Lagrange method. Further confirmation of the validity of this suggestion comes from the studies of the scaling limit of the fermionic number operator $\langle n_k \rangle_v$ in momentum space, which can be defined as follows:

$$\langle n_k \rangle_v = \sum_m e^{-ikm} \lim_{t \rightarrow \infty} \langle \varphi | c_{vI+m}^\dagger(t) c_{vI}(t) | \varphi \rangle. \quad (31)$$

This quantity can be calculated for the special case of $m_0 = 1/2$ (see Appendix B) and one finds that

$$\langle n_k \rangle_v = \begin{cases} 1 & \text{if } -\arccos(v) + \pi/2 < k < \arccos(v) + \pi/2, \\ 0 & \text{else.} \end{cases} \quad (32)$$

Since n_k is a projection operator with eigenvalues 0,1 on the eigenstates of H^{xx} this result shows that the system evolves locally [in the scaling sense described by Eq. (22)] into an eigenstate of the XX chain. The occupied levels are the same as those obtained by the Lagrange multiplier method for the given set of $(m, j^M, j^I=0)$. This can be interpreted as the local scaling states being equivalent to the ground states found in the Lagrange multiplier method.

In order to demonstrate the value of the equivalence suggested by the above considerations, let us show how simple is the derivation of the magnetization profile if we assume that the Lagrange multiplier method can be used to describe the scaling limit. The calculation is based on the observation that the locally flat magnetization [$m(n+1, t) - m(n, t) \sim t^{-1} \Phi' \rightarrow 0$] implies that one can take the continuum limit of the equation describing the conservation of magnetization

$$\partial_t m + \partial_x j^M = 0. \quad (33)$$

Next, using the fact that the energy density is constant in both space and time one finds the states that can be used to describe the scaling limit for a given initial magnetization. It follows from Eq. (30) that these states are located along $h = \text{const}$ lines in the j^E, h phase diagram as shown in Fig. 3. Along this path, the current j^M depends only on the local magnetization

$$j^M(x, t) = F(m) = \frac{1}{\pi} \sin[\pi(m_0 - |m|)], \quad (34)$$

where $F(m)$ is obtained from Eqs. (28) and (29) using the fact that h is given by the initial energy density through Eq. (30). It follows then that the continuity equation (33) has a scaling solution [including the solution $m=0$] and that the magnetization profile $m(x, t) = \Phi(x/t)$ can be obtained from the following equation:

$$\frac{dF}{dm} = \frac{x}{t}. \quad (35)$$

Taking into account the limiting behavior $\Phi(z \rightarrow \pm \infty) = \pm m_0$ the straightforward steps described above lead to the

results (14) obtained from a much more complicated calculation treating the dynamics of the system.

V. FINAL REMARKS

We have shown that in the zero-temperature XX model, an initial state with a steplike magnetization inhomogeneity evolves into a scaling state that is remarkable for its magnetization profile (emergence of a flat part) and for the existence of locally homogeneous, current carrying states. We have also provided evidence that the local states of the scaling limit can be obtained by the Lagrange multiplier method. The possibility of applying the Lagrange method would simplify the calculation of transport significantly, thus it would be important to test the generality of our results. The simplest questions to ask are whether the results could be extended to finite temperatures and whether the calculation could be generalized to the transport of energy as well. It appears that these questions can be answered and work is in progress along these lines.

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APPENDIX A: MAGNETIZATION PROFILE

We are considering the case $m_0 \neq 1/2$ and study the scaling limit of the discrete derivative $\Phi'_n(v)$ defined in Eq. (17):

$$\begin{aligned} \Phi'_n(v) &= t[m(n+1, t) - m(n, t)]_{n/t=v} \\ &= t \left[-\frac{2m_0\pi}{\pi} J_n^2(t) \right. \\ &\quad \left. - 2 \sum_{r=1}^{\infty} \frac{\sin(2m_0\pi r)}{2\pi r} [J_n(t)J_{n+2r}(t) \right. \\ &\quad \left. + J_{-n}(t)J_{-n+2r}(t)] \right]_{n/t=v}. \end{aligned} \quad (\text{A1})$$

In order to take the scaling limit, we divide the sum into three parts where the indexes of the Bessel functions are positive. Using the limiting properties of the Bessel functions, one obtains the following form ($n \rightarrow \infty$ and $0 < v < 1$):

$$\begin{aligned} \Phi'_{n \rightarrow \infty}(v) &= -\frac{4m_0\pi}{\pi^2\sqrt{1-v^2}} \cos^2\left(\frac{n}{v}\gamma(v) - \frac{\pi}{4}\right) \\ &\quad - 2\sqrt{\frac{2}{\pi\sqrt{1-v^2}}} \left(\sum_{r=1}^{\infty} (A_r^+ + A_r^-) \right. \\ &\quad \left. - [1 - (-1)^n] \sum_{r=[n/2]+1}^{\infty} A_r^- \right) \cos\left(\frac{n}{v}\gamma(v) - \frac{\pi}{4}\right) \end{aligned} \quad (\text{A2})$$

with

$$\begin{aligned} A_r^{\pm} &= \lim_{n \rightarrow \infty} \frac{\sin(2m_0\pi r)}{2\pi r} \cos\left[\frac{n}{v}\gamma\left(v \pm \frac{2rv}{n}\right) - \frac{\pi}{4}\right] \\ &\quad \times \sqrt{\frac{2}{\pi\sqrt{1-(v \pm 2rv/n)^2}}} \\ &= \frac{\sin(2m_0\pi r)}{2\pi r} \cos\left[\frac{n}{v}\gamma(v) - \frac{\pi}{4} - 2r \arccos v\right] \\ &\quad \times \sqrt{\frac{2}{\pi\sqrt{1-v^2}}}, \end{aligned} \quad (\text{A3})$$

where $\gamma(x) = \sqrt{1-x^2} - x \arccos x$. We can drop the second sum since it vanishes with $n \rightarrow \infty$. Using trigonometrical identities, the calculation is reduced to evaluating the following sums (for $0 < x < 2\pi$):

$$\sum_{r=1}^{\infty} \frac{\sin(rx)}{r} = \frac{\pi-x}{2},$$

and

$$\sum_{r=1}^{\infty} \frac{\cos(rx)}{r} = -\frac{1}{2} \ln[2(1-\cos x)] \quad (\text{A4})$$

and one finds

$$\Phi'_{n \rightarrow \infty}(v)$$

$$= \begin{cases} 0 & \text{for } v < \cos m_0\pi, \\ -\frac{2}{\pi\sqrt{1-v^2}} \cos^2\left(\frac{n}{v}\gamma(v) - \pi/4\right) & \text{for } v > \cos m_0\pi. \end{cases} \quad (\text{A5})$$

Just as in Sec. III B one can see that $\Phi' \neq \lim_{n \rightarrow \infty} \Phi'_n$ since the differentiation and the scaling limit cannot be exchanged. However, one can obtain Φ from Eq. (A5) by simply integrating it with the substitution $\cos^2 \rightarrow 1/2$ as it is a rapidly oscillating function of v in the scaling limit

$$\Phi(v) = \lim_{n \rightarrow \infty} \int_0^v \Phi'_n(y) dy = \begin{cases} 0 & \text{for } v < \cos m_0 \pi, \\ - \int_{\cos m_0 \pi}^v \frac{dy}{\pi \sqrt{1-y^2}} = -m_0 + \frac{1}{\pi} \arccos v & \text{for } v > \cos m_0 \pi. \end{cases} \quad (\text{A6})$$

APPENDIX B: FERMION OCCUPATION NUMBER IN THE SCALING LIMIT

In a translationally invariant system the expectation value of the fermionic number operator $\langle n_k \rangle = \langle b_k^\dagger b_k \rangle$ in momentum space is expressed in terms of the real-space creation and annihilation operators as the Fourier transform of the two-point correlation function

$$\langle n_k \rangle = \sum_m e^{-ikm} \langle c_{j+m}^\dagger c_j \rangle, \quad (\text{B1})$$

where the site number j is arbitrary (because of translational invariance). In order to investigate this expectation value in the scaling regime we choose $j = vt$ and take the stationary limit $t \rightarrow \infty$ in the expectation value *before* performing the summation over m . In this way we probe the scaling state of the system in the region of the shifted origin located in the point vt where the state becomes translationally invariant in the long-time limit.

At time t the correlation function $\langle c_{j+m}^\dagger c_j \rangle$ for the initial state with $m_0 = 1/2$ is obtained from Eq. (4) as

$$\begin{aligned} \langle c_{j+m}^\dagger c_j \rangle &= (-i)^m \sum_{s=0}^{\infty} J_{s+m+j}(t) J_{s+j}(t) \\ &= (-i)^m J_{m+j} J_j + \frac{(-i)^m t}{2j} (J_{m+j} J_{j+1} - J_{m+j+1} J_j). \end{aligned} \quad (\text{B2})$$

Using the asymptotic expression of the Bessel functions [10]

$$J_{vt+m}(t) \sim \sqrt{\frac{2}{\pi t \sqrt{1-v^2}}} \cos \left(t \sqrt{1-v^2} - (vt+m)\alpha - \frac{\pi}{4} \right) \quad (\text{B3})$$

with $\alpha = \arccos v$ and utilizing the addition theorems for trigonometric functions we thus obtain

$$\lim_{t \rightarrow \infty} \langle c_{vt+m}^\dagger(t) c_{vt}(t) \rangle = \frac{i^m}{\pi m} \sin \alpha m = \frac{1}{2\pi} \int_{-\alpha+\pi/2}^{\alpha+\pi/2} dp e^{ipm}. \quad (\text{B4})$$

The Fourier transformation finally yields

$$\langle n_k \rangle_v = \frac{1}{2\pi} \int_{-\alpha+\pi/2}^{\alpha+\pi/2} dp \sum_m e^{i(p-k)m} = \int_{-\alpha+\pi/2}^{\alpha+\pi/2} dp \delta(p-k) \quad (\text{B5})$$

and hence Eq. (32).

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[11] It should be noted that although $\Phi'(v) \neq \lim_{n \rightarrow \infty} \Phi'_n(v)$ the integral of $\Phi'_n(v)$ does converge to $\Phi(v)$.